COMMENTS

Comments are short papers which criticize or correct papers of other authors previously published in the **Physical Review.** *Each Comment should state clearly to which paper it refers and must be accompanied by a brief abstract. The same publication schedule as for regular articles is followed, and page proofs are sent to authors.*

Reply to ''Comment on 'Exact solution of the wave dynamics of a particle bouncing chaotically on a periodically oscillating-wall' ''

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A reexamination of the results of my earlier paper [Phys. Rev. E **50**, 3116 (1994)] is supplied using entirely straightforward methods, i.e., without appealing to operator formalism. This reexamination confirms the consistency of the work in that paper by extracting a term proportional to time that Dembrinski *et al.* [Phys. Rev. E **53**, 4243 (1996)] claim is not present. The arguments of Dembinski *et al.* are not refuted, but it is pointed out that they are based upon methods within which one readily encounters paradoxes such $[\hat{p}^2,\hat{p}]\neq 0$ and $[\hat{x},\hat{p}]\neq\hat{i}$. The present exposition avoids such methods. Unfortunately I am not prepared to address their numerical results. This will have to await further investigation. However, an example of numerical results of my own is presented in order to illustrate the richness of the dynamics of the model. [S1063-651X(96)10306-8]

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The Comment of Dembinski *et al.* [1] is a serious and considered objection to the results of my earlier paper [2], hereafter referred to as QCB (quantized chaotic bouncer). I am in total agreement with the remarks of these authors that the formulation of a quantum problem on a half space (or half line in this case) remains incomplete to the best of our collective knowledge. Physically, of course, the reason is that the coordinate space representative of the momentum operator generates translations—if one is not careful one may inadvertently postulate the existence of translations that take one outside of the half space. The view expressed in QCB is that although the natural basis for consideration of such problems would seem to be the coordination representation, since it is there that the notion of ''half space'' is intuitively obvious, in fact that may be the wrong thing to do.

This alternative view is supported by the brief discussion cited in QCB from von Neumann's book [3]. Succinctly the argument is this. Suppose that the following statements are all true:

$$\hat{x}|x\rangle=x|x\rangle;\quad \hat{1}=\int_0^\infty dx|x\rangle\langle x|,\quad \langle x|x'\rangle=\delta(x-x');$$

$$\hat{p}|k\rangle=k|k\rangle;\quad \hat{1}=\int_{-\infty}^\infty dk|k\rangle\langle k|;\quad \langle k|k\rangle=\delta(k-k').$$

But since $\langle x|k\rangle=(1/\sqrt{2\pi})e^{ikx}$ (as reviewed in Sec. III A of QCB, without invoking completeness of the coordinate eigenstates) the six statements are incompatible. One way out of the conundrum is to abandon completeness of the coordinate eigenstates. While this does not *require* primary emphasis on the momentum eigenstates, it certainly suggests it.

This has specific consequences that are relevant to the objections raised by Dembinski *et al.* For as they note, the crux of the analytic part of the objection is based upon Eq. (3.16) of Ref. [4], hereafter referred to as SS, which may be written as

$$-i(E_m-E_n)\int_0^\infty dx\,\phi_m(x)\phi_n'(x)=ig(\delta_{n,m}-1). \qquad (1)$$

This result is used to claim that no term growing with the time t can be present in the partial derivative of the solution with respect to time. But such a term is present in QCB by virtue of an overall time-dependent phase, QCB Eq. (7). Thus it is argued that QBC cannot be correct.

The phase in question was introduced precisely to cancel a term proportional to t arising from the expression [QCB Eq. (20)]

$$\langle\phi_n|\hat{p}|\phi(t)\rangle=e^{-i\xi(t)}e^{-iE_nt}\int_{-\infty}^\infty dk\,k\,\phi_n^*(k)e^{-ih(t)k}\langle k|\phi(0)\rangle - gt\langle\phi_n|\phi(t)\rangle,$$

which manifestly contains a term proportional to the time t. The key issue is that this result was found to be a consequence of the canonical commutation relation $[\hat{x},\hat{p}]=i\hat{1}$ *when the left-hand side is evaluated using the momentum representation.*

The relevance to the arguments of Dembinski *et al.* is that Eq. (1) is a result of a calculation in the *coordinate* representation. To see the difference clearly, if the Hamiltonian $\hat{H}_0=\hat{p}^2/2+g\hat{x}$ has eigenstates $|n\rangle$, i.e., $H_0|n\rangle=E_n|n\rangle$, and if further $[\hat{p}^2,\hat{p}]=0$ one expects

$$\langle n|[\hat{H}_0,\hat{p}]|m = (E_n - E_m)\langle n|\hat{p}|m\rangle = g\langle n|[\hat{x},\hat{p}]|m\rangle = ig\,\delta_{n,m}$$

rather than Eq. (1).

This notwithstanding, Eq. (1) can be verified by elementary methods using the coordinate representation for the energy eigenstates. The ''extra term'' on the right-hand side (RHS) of that equation, $-ig$, arises from the identity

$$(A,B'') = (A'',B) + A'(0)B(0) - A(0)B'(0),$$

with (A,B) the integral of $A(x)B(x)$ over the positive half space, together with properties of $\partial_x\phi_n(x)|_{x=0}$ [4].

Thus if $A(x) = \phi_n(x)$ and $B(x) = \partial_x\phi_n(x)$, the above identity suggests that $\hat{p}$ is a nonassociative operator, $[\hat{p}^2,\hat{p}] \neq 0$. And to further appreciate how profoundly confusing these matters can be, it is relevant to note that SS *explicitly cite* in their Eq. (3.17) that $\langle m|[\hat{x},\hat{p}]|n\rangle$ has nondiagonal elements in coordinate representation based upon the exact methods the Comment cites. This is truly odd and deserves closer scrutiny. For while an arbitrary operator can certainly be diagonal in one basis and nondiagonal in another, the unit operator is by definition diagonal in all orthonormal bases, such as the eigenstates of $\hat{H}_0$. One cannot proceed to discuss the system as quantal without trepidation if the canonical commutation relation appears to be violated. What is going on?

It was argued in QCB that such conundrums arise from an erroneous insertion of incomplete coordinate eigenstates into the formal commutation relations. An analysis carried out in detail in QCB found that such difficulties already arise at the level of the free particle constrained to the half space $(g=0)$. They are not at all a consequence of the term $f(t)\hat{p}$ (analogous to a magnetic coupling term) present in the Hamiltonian describing the system under discussion.

The true ''problem'' is that equations such as Eq. (1) above must be considered as holding true among distributions, or generalized functions, rather than taken at face value. That is, their correctness depends upon what other functions they ''act upon'' within integrals or sums that define observables. A simple example is failure of completeness of the $\phi_n(x)$ when applied to functions that do not vanish at the origin.

The work in QCB also suggested a method to avoid apparent paradoxes such as those just noted ($[\hat{x},\hat{p}] \neq i\hat{1}$, $[\hat{p}^2,\hat{p}] = 0$). Concretely, QCB suggested that a ''good'' representation of the quantum operators should respect the canonical commutation relations. Thus it is prudent to work in momentum representation rather than coordinate representations so, e.g., $[\hat{p}^2,\hat{p}] = 0$ trivally. Work in quantum chaos is replete with attempts to find best bases within which to formulate the idiosyncracies of the specific problems being addressed. The proposed approach is hardly radical.

It is neither expected nor intended that the above discussion satisfactorily resolves the problems with quantization in the half space that Dembinski *et al.* and this author agree exist. But that does not matter. It will now be shown that the solution presented in QCB contains the time dependence corresponding to that of QCB Eq. (20), i.e., that which Dembinski *et al.* claim is absent.

Dembinski *et al.* claim that a certain term cannot have an unbounded time dependence based upon properties of the solution that was presented. The claim is not based on general principles. Thus the task at hand is to exhibit that the crucial time dependence is indeed present in the solution, contrary to the assertions of Dembinski *et al.* The approach will be to reexpress the solution using simple transformations of variables, without appealing to any operator relationships. The underlying philosophy reflects a dictum of Gell-Mann [5]: ''We may compare this process to a method sometimes employed in French cuisine: a piece of pheasant meat is cooked between two pieces of veal, which are then discarded.'' In the present context, although operator relationships were invoked to motivate the solution in QCB, the response to the Comment does not require their introduction. Thus also we avoid the task of interpreting how Eq. (1) is to be applied as a distribution.

The solution presented in QCB is [QCB, Eq. (25)]:

$$\phi(x,t) = \exp\left(ig\int_0^t dt'\; t'f(t')\right)\sum_n \phi_n(x)e^{-iE_nt} \times \int_0^\infty dx'\,\phi[x'-h(t),0]\phi_n(x'). \tag{2}$$

Here the $\phi_n(x)$ are the eigenstates of $\hat{H}_0$ and $\phi(x,t)$ is claimed to be a solution of the equation [QCB, Eq. (3)]

$$i\partial_t\phi(x,t) = -\frac{1}{2}\partial_x^2\phi(x,t) + gx\phi(x,t) - if(t)\partial_x\phi(x,t) \tag{3}$$

subject to the boundary condition $\phi(x=0,t)=0$ and otherwise arbitrary initial condition $\phi(x,0)$. In the above $h(t)=\int_0^t dt' f(t')$.

Since the $\phi_n(x)$ are real we may introduce the well-known representation

$$\phi_n(x) = \mathcal{N}_n\int dq\,\exp(-iq^3/6g)\exp\{iq(x_n-x)\},$$

in which $x_n = E_n/g$. Further, since the wave functions vanish for nonpositive argument, we may rearrange Eq. (2) into the form

$$\phi(x,t) = \exp\left(ig\int_0^t dt'\; t'f(t')\right)\sum_n \mathcal{N}_n\phi_n(x)e^{-iE_nt}\int_0^\infty dx'\,\phi(x',0)\int dq\,\exp(-iq^3/6g)\exp\{iq(x_n-h(t)-x')\}. \tag{4}$$

For convenience introduce the notation

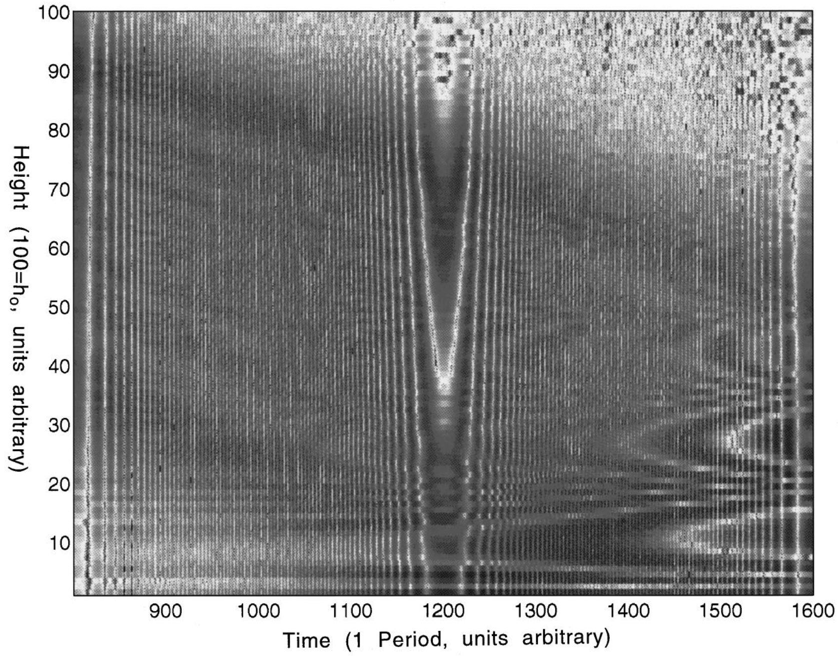

FIG. 1. Logarithm of the field intensity for $g=\omega=1$, $h_0=5$, and initial state mode 800, with dark regions most intense. The abscissa is the height, with the top of the graph slightly below h_0.

$$\Psi(k)=\exp(-ik^3/6g)\int_0^\infty dx'\,\phi(x',0)e^{-ikx'}.$$

Noting that $f(t)=dh/dt$, $\phi(x,t)$ may be further rewritten as

$$\begin{aligned}\phi(x,t)&=\exp\left(-ig\int_0^t dt'h(t')\right)\sum_n \mathcal{N}_n\phi_n(x)\int dq\,\exp\{i(q-gt)[x_n h(t)]\}\Psi(q)\\ &=\exp\left(-ig\int_0^t dt'\,h(t')\right)\sum_n \mathcal{N}_n\phi_n(x)\int dq\,\exp\{iq[x_n-h(t)]\}\Psi(q+gt)\\ &=\sum_n \phi_n(x)\int dq\,\tilde{\phi}_n^*(q)\exp[-iX(q,t)]\Psi(q+gt),\end{aligned}\tag{5}$$

with $X(q,t)\equiv -q^3/6g+qh(t)+g\int_0^t dt'\,h(t')$, and with the further definition $\tilde{\phi}_n(q)=\mathcal{N}_n\exp(iq^3/6g)\exp(-iqx_n)$.

The above intricate rewrite is motivated by the following:

$$i\partial_t[\exp[-iX(q,t)]\Psi(q+gt)] =\left[\frac{q^2}{2}+qf(t)+ig\partial_q\right][\exp[-iX(q,t)]\Psi(q+gt)].\tag{6}$$

By straightforward evaulation the function $\exp[-iX(q,t)]$ is itself a solution of Eq. (6). The first order structure of the derivative operators implies that Eq. (6) is satisfied for *any* (reasonable) function $\Psi(q+gt)$. Evidently the expression for Ψ implied by QCB is a particular choice designed for the full solution to collapse to the initial condition at $t=0$.

Now, in accord with Dembinski *et al.*, if $i\partial_t\phi(x,t)$ contains a term proportional to $tf(t)$, as follows from Eq. (4), that term must be canceled by a term in the RHS of Eq. (6). So we must examine the expression that has $f(t)$ as a coefficient,

$$\mathcal{I}=\int dq\,q\,\tilde{\phi}_n^*(q)\exp[-iX(q,t)]\Psi(q+gt).\tag{7}$$

But by simply undoing the series of manipulations that led to Eq. (5),

$$\mathcal{I}=\mathcal{N}_n\exp\left(ig\int_0^t dt'\,t'f(t')\right)e^{-iE_nt} \times\int dq(q-gt)\exp\{iq[x_n-h(t)]\}\Psi(q).\tag{8}$$

Rearrangement of the term in $\mathcal{I}\propto gt$ leads to

$$i\partial_t\phi(x,t)\supset -gt\,f(t)\sum_n \phi_n(x) \times\int dq\,\tilde{\phi}_n^*(q)\exp[-iX(q,t)]\Psi(q+gt).$$

Now, the above result is a direct consequence of rewriting the RHS of Eq. (6). Note that it was not necessary to use the specific forms for Ψ and X in order to achieve it. Without doubt, Eq. (8) contains a factor (gt) in the term that corresponds to the one into which Dembinski *et al.* inserted Eq. (12). To complete the discussion, it is only necessary to show that the rewrites that have been performed do not remove the phase term, i.e., the term corresponding to $i\partial_t \exp[ig\int_0^t dt' t' f(t')]$. This piece is readily achieved by evaluating the explicit expressions for $\partial_t X(q,t)$ and $|\partial_t \Psi(k+gt)$ without invoking the identity $\partial_t \Psi(k+gt) = g\partial_k \Psi(k+gt)$. That is, an ''explicit'' term (the phase-related expression) cancels the ''implicit'' term (related to matrix elements of $\hat{p}$) just as was claimed in QCB.

There is no question that, as mentioned earlier, Eq. (1) is an entirely correct result. According to the above, the results in QCB are consistent in spite of Eq. (1). It appears that the difference between the results of Dembinski *et al.* and those in QCB come about from different ways in which formal statements are implemented—that is, how the *content* of Eq. (1) is expressed within the entire context of the problem at hand.

To conclude this response, I have no comment regarding the numerical calculations performed by Dembinski *et al.* at the present time. For obvious reasons, numerical calculations must be carefully programmed, and while there is no reason to doubt the accuracy of their results out of hand, it will take more time to reproduce their work. I myself have performed independent numerical work based upon the asymptotics described in QCB. Figure 1 illustrates the behavior of the intensity just in the vicinity of a bounce, based upon the initial condition that $\phi(x,0)$ is an eigenmode of H_0. The main point of the figure (which, together with its companions, I hope to publish elsewhere) is to allay the concerns of Dembinski *et al.* that their Eq. (11) somehow implies that the quantized behavior is trivial. In fact it is not, and the behavior near the upper turning points is very interesting indeed. It is also clear from the figures in QCB that although the periodicity of the modal transition operator should be strictly true, a large number of modes couple to the initial mode almost immediately after the periodic point, rendering accurate numerical calculations delicate.

[1] S. T. Dembinski *et al.*, Phys. Rev. E **53**, 4243 (1996).
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[5] M. Gell-Mann, Physics **1**, 63 (1964).